Beta-function in QCD and gluon condensate

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Abstract

Under assumption of singular behavior of $\alpha_s(q^2)$ at $q^2 \simeq 0$ and of large q^2 behavior, corresponding to the perturbation theory up to four loops, a procedure is considered of matching the β-function at a boundary of perturbative and non-perturbative regions. The contribution of the non-perturbative region to the gluon condensate is calculated with varying normalization condition $\alpha_s(m_\tau^2) = 0.29, 0.30, ..., 0.36$ for two different ways of definition of the non-perturbative invariant charge in the infrared region. We obtain quite consistent results for values of the gluon condensate, nonperturbative region scale q_0, and the string tension σ.

It is well-known, that the β-function in the perturbative QCD is of the form

$$\beta_{\rm pert}(h) = -b_0 h^2 - 2b_1 h^3 - \frac{b_2}{2} h^4 - b_3 h^5 + O(h^6), \;\; h = \frac{\alpha_s(q^2)}{4\pi}. \tag{1}$$

For $n_f = 3$ we have values of coefficients $b_0 = 9$, $b_1 = 32$, $b_2 \simeq 1287.67$, $b_3 \simeq 12090.38$ (coefficients b_0, b_1 do not depend on renormalization scheme while values b_2, b_3 correspond to a choice of $\overline{MS}$-scheme). Expressions obtained by solution of Gell-Mann – Low equation

$$q^2 \frac{\partial h(q^2)}{\partial q^2} = \beta(h) \tag{2}$$

with the use of (1), are widely used for sufficiently large momenta transfer, however they can not be applied in the infrared region. For a behavior of the invariant charge (the running coupling constant) $\alpha_s(q^2)$ at $q^2 \to 0$ a number of variants are considered (see, e.g., [1]). In particular singular infrared asymptotic at $q^2 = 0$

$$\alpha_s(q^2) \simeq \frac{g^2}{4\pi}\frac{M^2}{q^2}, \quad q^2 \to 0 \tag{3}$$

is possible (see, e.g., review [2] and more recent papers [3]). Such behavior naturally corresponds to a linear confining quark-antiquark static potential at that $g^2M^2 = 6\pi\sigma$, where σ is the string tension. Results of some works on the lattice study of the three-gluon vertex [4] demonstrate a necessity of taking into account of non-perturbative contributions to the running coupling constant being of the form of (3). In the framework of the continuous QFT arguments in favor of behavior (3) are also presented in recent paper [5].

Asymptotic behavior (3) occurs provided $\beta(h) \to -h$ for $h \to \infty$. Let us consider a possibility of behavior (3) to be valid and assume the following form of the infrared β-function

$$\beta(h) = -h + z, \quad h > h_0, \tag{4}$$

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where z is a constant and h_0 defines the boundary between perturbative and non-perturbative regions. For $h < h_0$ we shall use β-function (1) with a finite number of loops taken into account. Our recipe for construction of β-function for all $h > 0$ consists in a smooth matching of expressions (4) and (1) at point $h = h_0$ in approximations of the perturbation theory up to four loops. The demand of the β-function and its derivative to be continuous uniquely fix free parameters z and h_0 of the "global" β-function (the matched one).

For an illustration let us consider the most simple one-loop case. Conditions of matching give two equations

$$\begin{aligned} -b_0 h_0^2 &= -h_0 + z, \\ -2b_0 h_0 &= -1. \end{aligned} \tag{5}$$

The solution of set (5) reads

$$h_0 = \frac{1}{2b_0}, \quad z = \frac{1}{4b_0}. \tag{6}$$

We shall normalize perturbative solution

$$\alpha_s(q^2) = \frac{4\pi}{b_0 \ln x}, \quad x = \frac{q^2}{\Lambda_{QCD}^2}, \quad q^2 \geq q_0^2 \tag{7}$$

by value $4\pi h_0$, that gives

$$x_0 = q_0^2/\Lambda_{QCD}^2 = e^2, \tag{8}$$

where $e = 2.71828...$. Imposing on $\alpha_s(q^2)$ the natural condition to be continuous at $q^2 = q_0^2$, we may normalize non-perturbative solution of equation (2)

$$\alpha_s(q^2) = 4\pi \left(\frac{C}{q^2} + z \right), \quad q^2 \leq q_0^2 \tag{9}$$

by $4\pi h_0$ as well. As a result we obtain

$$C = q_0^2(h_0 - z), \quad c_0 \equiv C/\Lambda_{QCD}^2 = x_0(h_0 - z) = \frac{e^2}{4b_0}. \tag{10}$$

For final fixation of the solution for all $q^2 > 0$ we need to define Λ_{QCD} by normalizing the solution, say, at point $q^2 = m_\tau^2$, where $m_\tau = 1.77703$ GeV is the mass of the τ-lepton [6].

Let us turn to calculation of the gluon condensate. Its value is defined by the non-perturbative part of α_s. We have (see, e.g., the third of refs. [3])

$$K \equiv < \alpha_s/\pi : G^a_{\mu\nu} G^a_{\mu\nu} :> = \frac{3}{\pi^3} \int\limits_0^\infty dq^2\, q^2 \alpha_{\rm npt}(q^2) = \frac{3}{\pi^3} \int\limits_0^\infty dq^2\, q^2 (\alpha_s(q^2) - \alpha_{\rm pert}(q^2)). \tag{11}$$

In our approach the non-perturbative contribution is present for $q^2 < q_0^2$ only. For the beginning we define the perturbative part at this region basing on the assumption of freezing of α_s at small q^2 (see, e.g., [7]). That is we assume

$$\alpha_{\rm pert}(q^2) \;=\; \alpha_s(q_0^2) \;=\; 4\pi h_0, \;\; q^2 < q_0^2. \tag{12}$$

Using expressions (9), (12), we have

$$\begin{aligned} K = \frac{12}{\pi^2} \int\limits_0^{q_0^2} dq^2\, q^2 \left(\frac{C}{q^2} + z - h_0 \right) = \frac{12}{\pi^2} q_0^4 \left(\frac{C}{q_0^2} + \frac{1}{2}(z - h_0) \right) = \\ = \frac{6}{\pi^2}(h_0 - z) x_0^2 \Lambda_{QCD}^4. \end{aligned} \tag{13}$$

Expression (13) is valid for each of 1 — 4-loop approximations of the perturbative α_s and ratio K/Λ^4_{QCD} does not depend on a normalization of α_s. In the one-loop case we have from (6), (8), (13) with $b_0 = 9$

$$K = \frac{e^4}{6\pi^2}\Lambda^4_{QCD}. \tag{14}$$

In the one-loop case normalization condition $\alpha_s(m_\tau^2) = 0.32$ gives $\Lambda_{QCD} = 0.201$ GeV, and expression (14) gives $K = (0.197 \text{ GeV})^4$, which is far from the conventional value of the gluon condensate (0.33 ± 0.01 GeV)^4 [8].

Let us consider multi-loop cases. Solution $h(q^2)$ of equation (2) for $L = \ln(q^2/\Lambda^2) \to \infty$ reads as follows

$$\begin{aligned} h(q^2) = \frac{1}{b_0 L}\Bigg\{ & 1 - \frac{2b_1}{b_0^2 L}\ln L + \frac{4b_1^2}{b_0^4 L^2}\left[\ln^2 L - \ln L - 1 + \frac{b_0 b_2}{8b_1^2}\right] \\ & - \frac{8b_1^3}{b_0^6 L^3}\left[\ln^3 L - \frac{5}{2}\ln^2 L - \left(2 - \frac{3b_0 b_2}{8b_1^2}\right)\ln L \right. \\ & \left. + \frac{1}{2} - \frac{b_0^2 b_3}{16 b_1^3}\right] + O\left(\frac{1}{L^4}\right)\Bigg\}. \end{aligned} \tag{15}$$

Keeping in the expression terms with powers of logarithms in denominators up to the first, the second, the third and the fourth, we fix the 1 — 4-loop approximations of the perturbation theory for running coupling constant. It may be written in the form

$$\alpha_s(q^2) = 4\pi h(q^2) = \frac{4\pi}{b_0} a(x),$$

$$\begin{aligned} a(x) = & \frac{1}{\ln x} - b\frac{\ln(\ln x)}{\ln^2 x} + b^2\left[\frac{\ln^2(\ln x)}{\ln^3 x} - \frac{\ln(\ln x)}{\ln^3 x} + \frac{\kappa}{\ln^3 x}\right] \\ & - b^3\left[\frac{\ln^3(\ln x)}{\ln^4 x} - \frac{5}{2}\frac{\ln^2(\ln x)}{\ln^4 x} + (3\kappa + 1)\frac{\ln(\ln x)}{\ln^4 x} + \frac{\bar\kappa}{\ln^4 x}\right]. \end{aligned} \tag{16}$$

Here $x = q^2/\Lambda^2$, and coefficient are defined as follows

$$\begin{aligned} b &= \frac{2b_1}{b_0^2}, \\ \kappa &= -1 + \frac{b_0 b_2}{8b_1^2}, \\ \bar\kappa &= \frac{1}{2} - \frac{b_0^2 b_3}{16 b_1^3}. \end{aligned} \tag{17}$$

Coefficients b, κ, $\bar\kappa$ depend on n_f.With $n_f = 3$ we have $b \simeq 0.7901$, $\kappa \simeq 0.4147$, $\bar\kappa \simeq -1.3679$. In the case of the two-loop approximation for perturbative α_s we have the following set of equations

$$\begin{gathered} b_0 h_0^2 + 2b_1 h_0^3 = h_0 - z, \\ 2b_0 h_0 + 6b_1 h_0^2 = 1, \\ \frac{1}{\ln x_0} - b\frac{\ln(\ln x_0)}{\ln^2 x_0} = b_0 h_0, \\ \frac{c_0}{x_0} + z = h_0. \end{gathered} \tag{18}$$

The set for three loops reads

$$\begin{gathered} b_0 h_0^2 + 2b_1 h_0^3 + \frac{b_2}{2}h_0^4 = h_0 - z, \\ 2b_0 h_0 + 6b_1 h_0^2 + 2b_2 h_0^3 = 1, \end{gathered}$$

$$\frac{1}{\ln x_0} - b\frac{\ln(\ln x_0)}{\ln^2 x_0} + b^2\left[\frac{\ln^2(\ln x_0)}{\ln^3 x_0} - \frac{\ln(\ln x_0)}{\ln^3 x_0} + \frac{\kappa}{\ln^3 x_0}\right] = b_0 h_0, \tag{19}$$
$$\frac{c_0}{x_0} + z = h_0.$$

The set for four loops reads

$$b_0 h_0^2 + 2b_1 h_0^3 + \frac{b_2}{2}h_0^4 + b_3 h_0^5 = h_0 - z,$$
$$2b_0 h_0 + 6b_1 h_0^2 + 2b_2 h_0^3 + 5b_3 h_0^4 = 1,$$
$$\frac{1}{\ln x_0} - b\frac{\ln(\ln x_0)}{\ln^2 x_0} + b^2\left[\frac{\ln^2(\ln x_0)}{\ln^3 x_0} - \frac{\ln(\ln x_0)}{\ln^3 x_0} + \frac{\kappa}{\ln^3 x_0}\right]$$
$$-b^3\left[\frac{\ln^3(\ln x_0)}{\ln^4 x_0} - \frac{5}{2}\frac{\ln^2(\ln x_0)}{\ln^4 x_0} + (3\kappa+1)\frac{\ln(\ln x_0)}{\ln^4 x_0} + \frac{\bar{\kappa}}{\ln^4 x_0}\right] = b_0 h_0, \tag{20}$$
$$\frac{c_0}{x_0} + z = h_0.$$

From sets of equations (18) – (20) we find values of h_0, z, x_0 and $_0$. They are presented in Table 1. The value of $K^{1/4}/\Lambda_{QCD}$ calculated with the aid of expression (13) is also presented there. For the parameter of string tension σ by using expressions (9), (10) we have relation $\sigma/\Lambda_{QCD}^2 = 8\pi c_0/3$. Taking into account the existing data [6, 9, 10] we fix the momentum dependence of solutions for a number of

Table 1: The dimensionless parameters h_0, z, $x_0 = q_0^2/\Lambda_{QCD}^2$, $c_0 = C/\Lambda_{QCD}^2$, $K^{1/4}/\Lambda_{QCD}$ on the number of loops, $n_f = 3$

	1-loop	2-loop	3-loop	4-loop
h_0	0.0556	0.0392	0.0356	0.0337
z	0.0278	0.0215	0.0203	0.0197
x_0	7.3891	7.7763	10.2622	12.4305
c_0	0.2053	0.1374	0.1572	0.1741
$K^{1/4}/\Lambda_{QCD}$	0.9799	0.8977	0.9952	1.0709

values of the running coupling constant at τ-lepton mass scale m_τ with the effective number of flavors n_f being equal to 3. The corresponding to these normalization conditions values of Λ_{QCD} are presented in Table 2, values of boundary momentum $q_0 = \sqrt{x_0}\Lambda_{QCD}$ are presented in Table 3, values of the gluon condensate $K^{1/4}$ are presented in Table 4 and the string tension parameter σ is given in Table 5.

Table 2: Values of the parameter Λ_{QCD} (GeV) on loop numbers and normalization conditions. Normalization conditions: $\alpha_s(m_\tau^2) = 0.29, 0.30, ..., 0.36$ with $m_\tau = 1.77703$ GeV, $n_f = 3$

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.1600	0.3168	0.2873	0.2837
0.30	0.1734	0.3370	0.3069	0.3026
0.31	0.1869	0.3568	0.3263	0.3212
0.32	0.2005	0.3762	0.3454	0.3394
0.33	0.2143	0.3951	0.3642	0.3573
0.34	0.2280	0.4136	0.3827	0.3749
0.35	0.2418	0.4315	0.4007	0.3920
0.36	0.2556	0.4490	0.4184	0.4087

Table 3: Values of the parameter q_0 (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.4350	0.8833	0.9203	1.0002
0.30	0.4713	0.9397	0.9831	1.0667
0.31	0.5081	0.9949	1.0452	1.1323
0.32	0.5451	1.0490	1.1065	1.1967
0.33	0.5824	1.1018	1.1667	1.2598
0.34	0.6198	1.1533	1.2258	1.3216
0.35	0.6572	1.2034	1.2838	1.3820
0.36	0.6947	1.2522	1.3405	1.4409

Table 4: Values of the gluon condensate $K^{1/4}$ (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.1568	0.2844	0.2859	0.3038
0.30	0.1699	0.3025	0.3054	0.3240
0.31	0.1832	0.3203	0.3247	0.3439
0.32	0.1965	0.3377	0.3437	0.3635
0.33	0.2099	0.3547	0.3624	0.3827
0.34	0.2234	0.3713	0.3808	0.4014
0.35	0.2369	0.3874	0.3988	0.4198
0.36	0.2504	0.4031	0.4164	0.4377

Table 5: String tension parameter $a = \sqrt{\sigma}$ (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.2098	0.3398	0.3297	0.3426
0.30	0.2274	0.3615	0.3522	0.3654
0.31	0.2451	0.3827	0.3745	0.3878
0.32	0.2630	0.4035	0.3964	0.4099
0.33	0.2809	0.4239	0.4180	0.4315
0.34	0.2990	0.4437	0.4392	0.4527
0.35	0.3170	0.4629	0.4599	0.4733
0.36	0.3351	0.4817	0.4802	0.4935

Let us consider another variant of definition of the perturbative α_s behavior for $q^2 < q_0^2$. Namely instead of freezing (12) we shall assume “forced” analytic behavior of α_s in this region,

$$\alpha_{\rm pert}(q^2) \,=\, \alpha_{\rm an}(q^2), \quad q^2 < q_0^2. \tag{21}$$

The main ideas of the analytic approach in quantum field theory, which allows one to overcome difficulties, connected with nonphysical singularities in perturbative expressions, were proposed in works [11, 12]. The analytic approach is successfully applied to QCD [13]. The forced analytic running coupling constant is

defined by the spectral representation

$$a_{\rm an}(y) = \frac{1}{\pi}\int\limits_0^\infty \frac{d\sigma}{y+\sigma}\rho(\sigma), \tag{22}$$

where spectral density $\rho(\sigma) = \Im a_{\rm an}(-\sigma - i0) = \Im a(-\sigma - i0)$. For the perturbative solutions $a(x)$ of the form (16) the two-loop analytic running coupling constant and its non-perturbative part were studied in ref. [14], the three-loop case and the four-loop case where studied in refs. [15] and [16], respectively. Let us write the spectral density up to the four-loop case.

$$\rho^{(1)}(\sigma) = \frac{\pi}{t^2+\pi^2}, \tag{23}$$

$$\rho^{(2)}(\sigma) = \rho^{(1)}(\sigma) - \frac{b}{(t^2+\pi^2)^2}\left[2\pi t F_1(t) - \left(t^2-\pi^2\right)F_2(t)\right], \tag{24}$$

$$\begin{aligned}\rho^{(3)}(\sigma) = \rho^{(2)}(\sigma) + \frac{b^2}{(t^2+\pi^2)^3}&\left[\pi\left(3t^2-\pi^2\right)\left(F_1^2(t)-F_2^2(t)\right) - 2t\left(t^2-3\pi^2\right)F_1(t)F_2(t)\right.\\ &\left.-\pi\left(3t^2-\pi^2\right)F_1(t) + t\left(t^2-3\pi^2\right)F_2(t) + \pi\kappa\left(3t^2-\pi^2\right)\right],\end{aligned} \tag{25}$$

$$\begin{aligned}\rho^{(4)}(\sigma) = \rho^{(3)}(\sigma) - \frac{b^3}{(t^2+\pi^2)^4}&\left[\left(t^4-6\pi^2t^2+\pi^4\right)\left(F_2^3(t)-3F_1^2(t)F_2(t)\right)\right.\\ &+4\pi t\left(t^2-\pi^2\right)\left(F_1^3(t)-3F_1(t)F_2^2(t)\right) - 10\pi t\left(t^2-\pi^2\right)\left(F_1^2(t)-F_2^2(t)\right)\\ &+5\left(t^4-6\pi^2t^2+\pi^4\right)F_1(t)F_2(t) + 4\pi\left(1+3\kappa\right)t\left(t^2-\pi^2\right)F_1(t)\\ &\left.-\left(1+3\kappa\right)\left(t^4-6\pi^2t^2+\pi^4\right)F_2(t) + 4\pi\bar{\kappa}t\left(t^2-\pi^2\right)\right].\end{aligned} \tag{26}$$

Here $t = \ln(\sigma)$,

$$F_1(t) \equiv \frac{1}{2}\ln(t^2+\pi^2), \quad F_2(t) \equiv \arccos\frac{t}{\sqrt{t^2+\pi^2}}. \tag{27}$$

Solving the equation[1]

$$a_{\rm an}(y) = b_0 h_0, \tag{28}$$

we find values y_0 for $a_{\rm an}$, being defined by formulas (22) – (27) with the use of values h_0 obtained above. Further we find dimensionless quantity $\xi = (\Lambda_{\rm an}/\Lambda_{QCD})^2 = x_0/y_0$. Values of y_0 and ξ in dependence on the number of loops are presented in Table 6. In Table 7 values of $\Lambda_{\rm an} = q_0/\sqrt{y_0}$ are presented in dependence on the number of loops and on normalization conditions at $q^2 = m_\tau^2$.

Let us turn to the gluon condensate. In the considered method of definition of the perturbative part of α_s in the non-perturbative region we have

$$K_{\rm an} = \frac{3}{\pi^3}\int\limits_0^{q_0^2} dq^2\, q^2(\alpha_s(q^2) - \alpha_{\rm an}(q^2)) = \frac{12}{\pi^2}\int\limits_0^{q_0^2} dq^2\, q^2\left(\frac{C}{q^2} + z - \frac{1}{b_0}a_{\rm an}\left(\frac{q^2}{\Lambda_{\rm an}^2}\right)\right). \tag{29}$$

Taking into account the spectral representation (22) and performing integration in Eq. (29), we obtain

$$K_{\rm an} = \frac{12}{\pi^2}\left(Cq_0^2 + \frac{z}{2}q_0^4\right) - \frac{12\Lambda_{\rm an}^4}{\pi^3 b_0}\int\limits_0^\infty d\sigma\,\rho(\sigma)\left[y_0 - \sigma\ln\left(1+\frac{y_0}{\sigma}\right)\right]. \tag{30}$$

Substitution $\sigma = \exp(t)$ with the use of expressions (8) and (10) leads to the following representation of the gluon condensate

$$K_{\rm an} = \Lambda_{QCD}^4\left[\frac{12}{\pi^2}\left(h_0 - \frac{z}{2}\right)x_0^2 - \frac{12\xi^2}{\pi^3 b_0}\int\limits_{-\infty}^\infty dt\,\rho(t)\left\{y_0 e^t - e^{2t}\ln\left(1+y_0e^{-t}\right)\right\}\right]. \tag{31}$$

[1] While solving this equation it is convenient to use the method of ref. [15], in which $\alpha_{\rm an}^{\rm npt}(y)$ is represented as a series in inverse powers of y

The expression in the square brackets of (31) does not depend on values of $\alpha_s(m_\tau^2)$, values of the ratio $K_{\rm an}^{1/4}/\Lambda_{QCD}$, which are obtained by numerical integration in formula (31), are given in Table 6. In Table 7 and Table 8 values of the parameter $\Lambda_{\rm an}$ and of the gluon condensate $K_{\rm an}^{1/4}$ are presented, respectively.

Table 6: Dimensionless parameters y_0, ξ, $K_{\rm an}^{1/4}/\Lambda_{QCD}$ on loop numbers, $n_f = 3$

	1-loop	2-loop	3-loop	4-loop
y_0	1.0000	0.8299	2.2691	2.8710
ξ	7.3893	9.3702	4.5225	4.3297
$K_{\rm an}^{1/4}/\Lambda_{QCD}$	0.9373	0.8494	0.9353	1.0025

Table 7: Dependence of the parameter $\Lambda_{\rm an}$ (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.4350	0.9696	0.6109	0.5903
0.30	0.4714	1.0315	0.6526	0.6296
0.31	0.5081	1.0921	0.6939	0.6682
0.32	0.5451	1.1515	0.7345	0.7063
0.33	0.5824	1.2094	0.7745	0.7435
0.34	0.6198	1.2659	0.8138	0.7800
0.35	0.6572	1.3210	0.8522	0.8156
0.36	0.6947	1.3745	0.8899	0.8504

Table 8: Values of the gluon condensate $K_{\rm an}^{1/4}$ (GeV) in dependence on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.1500	0.2691	0.2687	0.2844
0.30	0.1625	0.2862	0.2870	0.3033
0.31	0.1752	0.3031	0.3052	0.3219
0.32	0.1880	0.3195	0.3230	0.3403
0.33	0.2008	0.3356	0.3406	0.3582
0.34	0.2137	0.3513	0.3579	0.3758
0.35	0.2266	0.3666	0.3748	0.3929
0.36	0.2395	0.3814	0.3914	0.4097

For cases from one loop up to four loops the behavior of the running coupling constant $\alpha_s(q^2)$ for all $q^2 > 0$ is shown in Fig. 1. Here the behavior of the analytic coupling constant $\alpha_{\rm an}(q^2)$ for $q^2 < q_0^2$, which defines the perturbative part of $\alpha_s(q^2)$ in this region, is also shown. In Fig. 2 in addition to $\alpha_s(q^2)$ its non-perturbative part $\alpha_{\rm npt}(q^2)$ is also shown, which turns to zero for $q^2 > q_0^2$. Normalization condition for all curves in Fig. 1, Fig. 2 is $\alpha_s(m_\tau^2) = 0.32$.

Starting from the well-known perturbative behavior of the β-function for small values of the coupling constant h and the behavior for large values of the coupling constant, which corresponds to the linear confinement, we have constructed the β-function for all $h > 0$. To control dependence of the results on the number of loops we simultaneously consider cases corresponding to perturbative β-function for 1 — 4 loops. While constructing the matched β-function we assume it to be smooth at the matching point, that

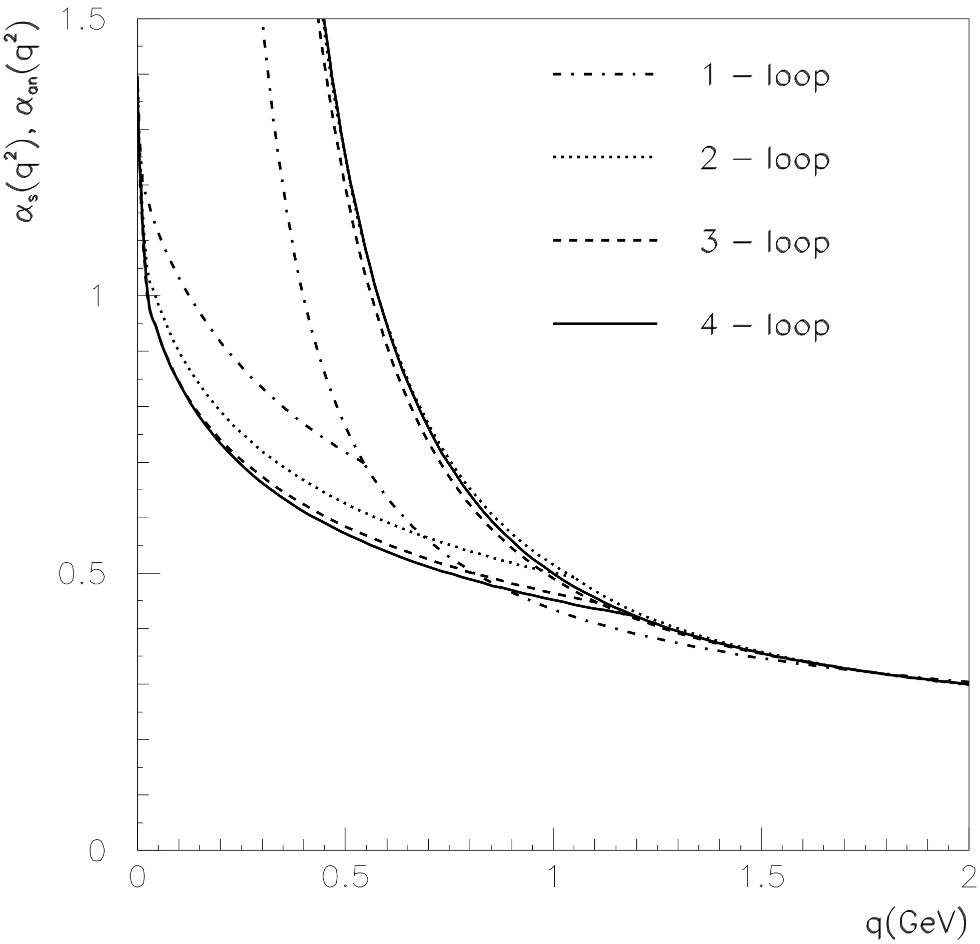

Figure 1: Running coupling constant $\alpha_s(q^2)$ for all $q^2 > 0$ and analytic coupling constant $\alpha_{\rm an}(q^2)$ for $q^2 < q_0^2$ (the corresponding curves are the lower ones). Normalization conditions: $\alpha_s(m_\tau^2) = 0.32$, $\alpha_{\rm an}(q_0^2) = \alpha_s(q_0^2)$

leads to the invariant charge being smooth together with its derivative for all $q^2 > 0$. The normalization of the invariant charge, e.g., at m_τ fix it thoroughly. Then value of $\alpha_s(m_\tau^2) \simeq 0.33$ corresponds to string tension $a \simeq 0.42$ GeV [17].

The obtained invariant charge is applied to study of a quite important physical quantity, the gluon condensate. In doing this we consider two variants of extracting of the non-perturbative contributions from the overall expression for the invariant charge. The first variant assumes "freezing" of perturbative part of the charge in the non-perturbative region $q^2 < q_0^2$, while for the second one we choose the analytic behavior of the perturbative part in this region. As we see from Tables 4, 8 the first variant leads to values of the gluon condensate being somewhat larger, than that for the second variant. Emphasize, that for $\alpha_s(m_\tau^2) = 0.33$ the gluon condensate for the second variant practically coincides with the conventional value [8], while for the first variant it turns to be $K^{1/4} \simeq 0.36$ GeV, that is rather higher, than the conventional value. Note, that other important non-perturbative parameter q_0 for the same normalization conditions also turns to be of reasonable magnitude, $q_0 \simeq 1.17$ GeV (Table 3). We may conclude,

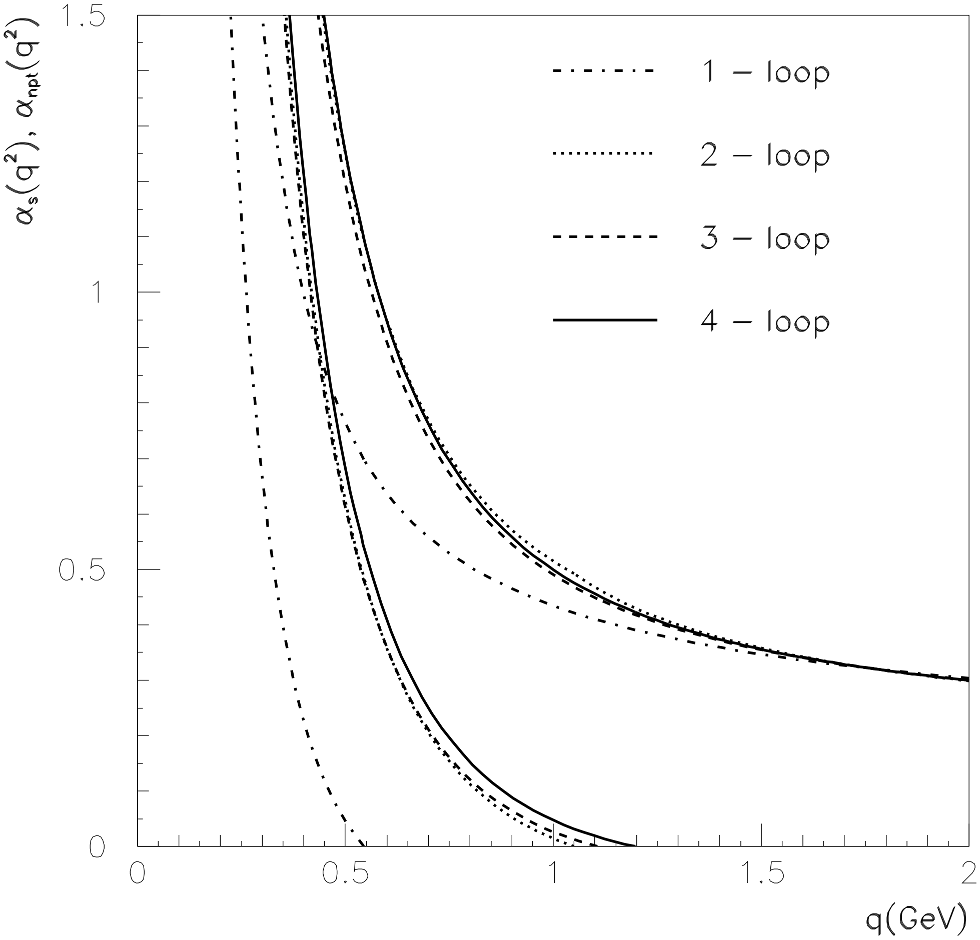

Figure 2: Running coupling constant $\alpha_s(q^2)$ and its non-perturbative part $\alpha_{\rm npt}(q^2)$ (the corresponding curves are the lower ones) with definition of $\alpha_{\rm pert}(q^2)$ by (21). Normalization condition: $\alpha_s(m_\tau^2) = 0.32$

that singular behavior of the running coupling constant (3) consistently describe these non-perturbative parameters.

The authors express deep gratitude to V.A. Petrov, V.E. Rochev and D.V. Shirkov for valuable discussions. The work has been partially supported by RFBR under Grant No. 02-01-00601.

References

[1] D.V. Shirkov, Teor. Mat. Fiz. **132**, 484 (2002) [Theor. Math. Phys. **132**, 1309 (2002)].

[2] B.A. Arbuzov, Fiz. Elem. Chast. Atom. Yad. **19**, 5 (1988) [Phys. Part. Nucl. **19**, 1 (1988)].

[3] A.I. Alekseev and B.A. Arbuzov, Yad. Fiz. **61**, 314 (1998) [Phys. At. Nucl. **61**, 264 (1998)]; A.I. Alekseev and B.A. Arbuzov, Mod. Phys. Lett. A, **13**, 1447 (1998); A.I. Alekseev, in *Proceedings of the Workshop on Non-perturbative Methods in Quantum Field Theory*, Adelaide, Australia, 1998,

edited by A.V. Schreiber, A.G. Williams and A.W. Thomas (World Scientific, Singapore, 1998), hep-ph/9808206.

[4] G. Burgio, F.Di Renzo, G. Marchesini and E. Onofri, Phys. Lett. B **422**, 219 (1998), Nucl. Phys. Proc. Suppl. **63**, 805 (1998); G. Burgio, F.Di Renzo, C. Parrinello and C. Pittori, Nucl. Phys. Proc. Suppl. **73**, 623 (1999), Nucl. Phys. Proc. Suppl. **74**, 388 (1999); Ph. Boucaud et al., JHEP **0004**, 006 (2000).

[5] V. Gogohia, Phys. Lett. B **584**, 225 (2004).

[6] Particle Data Group, D.E. Groom *et al.*, Eur. Phys. J. C **15**, 85 (2000).

[7] Yu.A. Simonov, Yad. Fiz. **58**, 113 (1995) [Phys. At. Nucl. **58**,107 (1995]; Yad. Fiz. **65**, 140 (2002).

[8] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B **147**, 385, 448 (1979).

[9] S. Bethke, J. Phys. G **26**, R27 (2000), hep-ex/0004021.

[10] A.A. Pivovarov, Preprint INR-TH-03-3, Moscow 2003, hep-ph/0301074.

[11] P.J. Redmond, Phys. Rev. **112**, 1404 (1958).

[12] N.N. Bogolubov, A.A. Logunov, and D.V. Shirkov, Zh. Éksp. Teor.Fiz. **37**, 805 (1959) [Sov. Phys. JETP **10**, 574 (1960)].

[13] I.L. Solovtsov and D.V. Shirkov, Teor. Mat. Fiz. **120**, 482 (1999) [Theor. Math. Phys. **120**, 1210 (1999)].

[14] A.I. Alekseev, Phys. Rev. D, **61**, 114005 (2000).

[15] A.I. Alekseev, Yad. Fiz. **65**, 1722 (2002) [Phys. At. Nucl. **65**,1678 (2002)].

[16] A.I. Alekseev, Few-Body Systems **32**, 193 (2003).

[17] L.D. Soloviev, Phys. Rev. D **58**, 035005 (1998); Phys. Rev. D **61**, 015009 (2000).